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Domination (totally) Dot-critical of Harary Graphs

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Abstract

A graph G is referred to as domination vertex critical if the removal of any vertex results in a reduction of the domination number. It is considered dot-critical (or totally dot-critical) if contracting any edge (or identifying any two vertices) leads to a decrease in the domination number. In this concise paper, we delve into the investigation of these properties and proceed to characterize the dot-critical and totally dot-critical attributes of Harary graphs.

Keywords: domination number; dot-critical; totally dot-critical; Harary graph.

1 Introduction

Throughout this paper, the graphs under consideration are assumed to be both simple and connected. Let's take a graph $G = (V, E)$, where $V = V(G)$ denotes its vertex set and $E = E(G)$ signifies its set of edges.

The graph G has an *order* denoted as $n = n(G) = |V|$. For any given vertex $v \in V$, its *open neighborhood* is the set $N(v) = u \in V|uv \in E$, and its *closed neighborhood* is $N[v] = N(v) \cup v$. The *degree* of vertex v is denoted as $d(v) = d_G(v) = |V(v)|$. The graph's *maximum degree* and *minimum degree* are denoted as $\Delta = \Delta(G)$ and $\delta = \delta(G)$. The $\Delta = \Delta(G)$ of a vertex in a graph is the highest number of edges incident to any single vertex, while the $\delta = \delta(G)$ is the smallest number of edges incident to a vertex. These values provide insights into the graph's connectivity and structure. For any definitions and notations not provided here, we refer readers to [10]. Graph theory finds applications in various fields including computer science (networking, algorithms, data structures), social network analysis, transportation and logistics (route optimization, network design), biology (protein interaction networks, genetic analysis), scheduling and project management, communication networks, recommendation systems, and many other domains where relationships and connections need to be modeled and analyzed.

A set of vertices S within a graph G is refereed to as a *dominating set* if each vertex in $G \setminus S$ share an adjacency with a vertex in S. When S possesses the smallest feasilble size among all dominating sets of G, it is termed a *minimum dominating set* (abbreviated MDS). The size of any MDS for G is termed the *domination number* of G, represented by $\gamma(G)$ [4]. In a broader context, we assert that a set $A \subseteq V(G)$ dominates another set $B \subseteq V(G)$ if each vertex in $B \setminus A$ is connected to a vertex within A.

Concerning the domination number, a vertex v within the graph G is considered critical if $\gamma(G - v) < \gamma(G)$. A graph G is denoted as *vertex-critical* if this critical property holds for every vertex in G. A novel critical condition for the domination number was introduced by Burton et al. [3]. A graph is classified as *domination dot-critical* (referred to as *dot-critical* hereafter) if contracting an edge between any two adjacent vertices decreases the domination number of the resulting graph. When the process of identifying any pair of vertices within G leads to a reduced domination number, G is termed *totally dot-critical*. Given a pair of vertices a and b within G, the graph resulting from their identification is denoted as $G.ab$. When we say that G is *k-vertex-critical*, *k-dot-critical*, or *totally k-dot-critical*, we imply that it possesses the specified property and that $\gamma(G) = k$. For more comprehensive information, please refer to the sources [2, 3, 8]. Recent papers on domination for Jahangir graphs and Roman domination number, kindly refer [1, 9].

When $k \leq n$, arrange n vertices uniformly in a circular layout. If k is even, construct $H_{k,n}$ by creating connections between each vertex and its nearest $k/2$ vertices in both directions around the circle. When k is odd and n is even, form $H_{k,n}$ by linking each vertex to its nearest $(k-1)/2$ neighbors in both directions, as well as to the vertex diametrically opposite. In each scenario, $H_{k,n}$ becomes a k-regular graph. In the case where both k and n are odd, assign indices to the vertices using integers modulo n. The construction of $H_{k,n}$ then continues by utilizing $H_{k-1,n}$ as a base, adding edges between vertices i and $i + (n-1)/2$ for every $0 \leq i \leq (n-1)/2$. The resulting graph, denoted as $H_{k,n}$, in each case, is referred to as a "Harary graph" [10]. These graphs have been employed to analyze complex systems like transportation networks and biological networks, offering insights into their structural properties and behaviors.

The research discussed revolves around the examination of domination number and domination criticality within the context of Harary graphs. These investigations have been thoroughly explored in the works of Khodkar et al. [6] and Mojdeh et al. [7]. The primary objective of these studies is to unravel the core principles governing domination in Harary graphs, shedding light on the minimal number of vertices necessary to effectively control the entire graph. Furthermore, an additional study by Kartal et al. [5] delves into the analysis of semitotal domination within Harary graphs. This concept concerns the domination of vertices, where each vertex and its adjacent neighbors collectively dominate the entire graph. Building upon the ongoing investigations, the current paper serves as a natural progression in the exploration of domination-related properties unique to Harary graphs. Specifically, this paper delves into the investigation of dot-criticality and totally dot-criticality concepts within this particular graph class. The primary objective is to understand the impact of modifying structural aspects of Harary graphs on their domination properties, particularly concerning edge contractions and vertex identification. In essence, this research contributes significantly to refining our comprehension of the intricate interplay between graph structures and domination concepts, focusing on the specialized domain of Harary graphs.

The following results are useful.

Lemma 1.1. [6] *The following statements hold.*

(*i*) $\gamma(H_{2m,n}) = \left\lceil \dfrac{n}{2m+1} \right\rceil$.

(*ii*) *Let* $n-(m+1) = (2m+2)t+r$, *where* $0 \leq r \leq 2m+1$. *Then,*

$$\gamma(H_{2m+1,2n}) = \begin{cases} \left\lceil \dfrac{n}{2m+1} \right\rceil + 1, & \text{if } 2 \leq r \leq m+1 \text{ and } t+r \geq m+1, \\ \left\lceil \dfrac{n}{2m+1} \right\rceil, & \text{otherwise.} \end{cases}$$

(*iii*) *Let* $n-(m+1) = (2m+2)t+r$, *where* $0 \leq r \leq 2m+1$. *Then,*

$$\gamma(H_{2m+1,2n+1}) = \begin{cases} \left\lceil \dfrac{n}{m+1} \right\rceil + 1, & \text{if } 2 \leq r \leq m+1 \text{ and } t+r \geq m, \\ \left\lceil \dfrac{n}{m+1} \right\rceil, & \text{otherwise.} \end{cases}$$

Lemma 1.2. [7] *The following statements hold.*

(*i*) *The graph* $H_{2m,n}$, *with* $n = (2m+1)t+r$ *and* $0 \leq r \leq 2m$, *is* γ*-critical if and only if* $r = 1$.

(*ii*) *The Harary graph* $H_{2m+1,2n}$, *with* $2n = (2t+1)(2m+2)+2r$ *and* $0 \leq r \leq 2m+1$, *is not* γ*-critical for the cases* $r = 0$, $m+2 \leq r \leq 2m+1$, $\big(2 \leq r \leq m+1$ *and* $1 \leq t+r \leq m\big)$ *and* $\big(3 \leq r \leq m+1$ *and* $t+r \geq m+1\big)$, *and it is* γ*-critical for the case* $\big(r = 2$ *and* $t+r \geq m+1\big)$.

(*iii*) *The Harary graph* $H_{2m+1,2n+1}$, *with* $2n+1 = (2t+1)(2m+2)+2r+1$ *and* $0 \leq r \leq 2m+1$, *is* γ*-critical if and only if* $2 \leq r \leq m+1$ *and* $t+r = m$.

Lemma 1.3. [2] *Let consider a graph* G *and vertices* a *and* b *belonging to its vertex set* $V(G)$. *The inequality* $\gamma(G.ab) < \gamma(G)$ *holds if and only if there exists a Minimum Dominating Set* (MDS) S *within* G *where both* a *and* b *are part of* S, *or if at least one of the vertices* a *or* b *is critical within* G.

Lemma 1.4. [2] *A graph* G *having a domination number* $\gamma(G) = k \geq 2$ *is categorized as dot-critical (or totally dot-critical) if and only if each pair of adjacent non-critical vertices (or any pair of non-critical vertices) is encompassed within a shared (MDS) minimum dominating set.*

2 (Totally) Dot-criticality of $H_{2m,n}$

In this section, we provide a comprehensive criterion that is both necessary and sufficient to ascertain whether Harary graphs of the $H_{2m,n}$ type exhibit (total) dot-critical characteristics. This criterion involves investigating conditions that define dot-criticality or total dot-criticality. The elucidation of the intricate connections between these graph structures and their domination attributes hinges on a pivotal lemma. For this purpose, the following lemma proves to be invaluable.

Lemma 2.1. *For the graph $H_{2m,n}$, if $n = (2m+1)t + r$ where $0 \leq r \leq 2m$ and $r \neq 1$, then i and $i+1$ do not belong to any common MDS.*

Proof. In view of Part (i) of Lemma 1.1, we have $\gamma(H_{2m,n}) = t$ for $r = 0$, and $\gamma(H_{2m,n}) = t+1$ otherwise. Suppose to the contrary that there exists an MDS S of $H_{2m,n}$ such that $i, i+1 \in S$. Obviously, $\{i, i+1\}$ dominates $2m+2$ vertices of $H_{2m,n}$. So, the vertices which are not dominated by $\{i, i+1\}$ must be dominated by $S \setminus \{i, i+1\}$ with $|S \setminus \{i, i+1\}| = \gamma(H_{2m,n}) - 2$.

Note that $S \setminus \{i, i+1\}$ dominates at most $(2m+1)\big(\gamma(H_{2m,n}) - 2\big)$ vertices of $H_{2m,n}$. Therefore, $n - (2m+2) \leq \big(\gamma(H_{2m,n}) - 2\big)(2m+1)$, a contradiction. □

Theorem 2.1. *Let $G = H_{2m,n}$ be a Harary graph with $n = (2m+1)t + r$ where $0 \leq r \leq 2m$. Then, $H_{2m,n}$ is (totally) dot-critical if and only if $r = 1$.*

Proof. Let G be a (totally) dot-critical graph. By Lemma 1.3, for each $i \in V(G)$, there exists an MDS S of G such that $i, i+1 \in S$ or at least one of i or $i+1$ is critical in G. Suppose to the contrary that $r \neq 1$. By Lemma 1.2 (i), G has no critical vertices. By Lemma 2.1, there is no MDS of G containing both i and $i+1$ of G. So, G is neither dot-critical nor totally dot-critical by invoking Lemma 1.3. This is a contradiction. Thus $n = (2m+1)t + 1$.

Conversely, let $n = (2m+1)t + 1$. Then, $H_{2m,n}$ is a critical graph by Lemma 1.2 (i). Lemma 1.4 now implies that $H_{2m,n}$ is a (totally) dot-critical graph. □

3 (Totally) Dot-criticality of $H_{2m,n}$

In this section, we provide a comprehensive criterion that is both necessary and sufficient to ascertain whether Harary graphs of the $H_{2m,n}$ type exhibit (total) dot-critical characteristics. This criterion involves investigating conditions that define dot-criticality or total dot-criticality. The elucidation of the intricate connections between these graph structures and their domination attributes hinges on a pivotal lemma. For this purpose, the following lemma proves to be invaluable.

Lemma 3.1. *For the graph $H_{2m,n}$, if $n = (2m+1)t + r$ where $0 \leq r \leq 2m$ and $r \neq 1$, then i and $i+1$ do not belong to any common MDS.*

Proof. In view of Part (i) of Lemma 1.1, we have $\gamma(H_{2m,n}) = t$ for $r = 0$, and $\gamma(H_{2m,n}) = t+1$ otherwise. Suppose to the contrary that there exists an MDS S of $H_{2m,n}$ such that $i, i+1 \in S$. Obviously, $\{i, i+1\}$ dominates $2m+2$ vertices of $H_{2m,n}$. So, the vertices which are not dominated by $\{i, i+1\}$ must be dominated by $S \setminus \{i, i+1\}$ with $|S \setminus \{i, i+1\}| = \gamma(H_{2m,n}) - 2$.

Note that $S \setminus \{i, i+1\}$ dominates at most $(2m+1)(\gamma(H_{2m,n}) - 2)$ vertices of $H_{2m,n}$. Therefore, $n - (2m+2) \leq (\gamma(H_{2m,n}) - 2)(2m+1)$, a contradiction. □

Theorem 3.1. *Let $G = H_{2m,n}$ be a Harary graph with $n = (2m+1)t + r$ where $0 \leq r \leq 2m$. Then, $H_{2m,n}$ is (totally) dot-critical if and only if $r = 1$.*

Proof. Let G be a (totally) dot-critical graph. By Lemma 1.3, for each $i \in V(G)$, there exists an MDS S of G such that $i, i+1 \in S$ or at least one of i or $i+1$ is critical in G. Suppose to the contrary that $r \neq 1$. By Lemma 1.2 (i), G has no critical vertices. By Lemma 3.1, there is no MDS of G containing both i and $i+1$ of G. So, G is neither dot-critical nor totally dot-critical by invoking Lemma 1.3. This is a contradiction. Thus $n = (2m+1)t + 1$.

Conversely, let $n = (2m+1)t+1$. Then, $H_{2m,n}$ is a critical graph by Lemma 1.2 (i). Lemma 1.4 now implies that $H_{2m,n}$ is a (totally) dot-critical graph. □

4 (Totally) Dot-criticality of $H_{2m+1,2n}$

In this section, domination dot-criticality and totally dot-criticality of graph $H_{2m+1,2n+1}$ are verified.

Lemma 4.1. *For the graph $H_{2m+1,2n+1}$, let $n-(m+1) = (2m+2)t+r$, $0 \leq r \leq 2m+1$. If $r \in \{0, 1\}$, then vertex n belongs to S for any MDS S.*

Proof. Note that the degree of each vertex in $H_{2m+1,2n+1}$ is $2m+1$, except for the vertex n of degree $2m+2$. Now let S be an MDS of $H_{2m+1,2n+1}$. If $r = 0$, then $|S| = 2t+1$. If S does not contain the vertex n, then the number of vertices dominated by S equals $(2m+2)(2t+1) < 2n+1 = (2m+2)(2t+1)+1$, a contradiction.

Let $r = 1$. Then $|S| = 2t+2$. If $n+1 \notin S$ then one of the vertices 1, $2n+1$ and v where $v \in \{n+1+m, \ldots, n+2, n, \ldots, n-m\}$ has to be in S. Since the vertices take place non-bilateral on the graph respected to vertex $n+1$ then we cannot $2t+2$ vertices for dominating set when $n+1 \notin S$. □

Theorem 4.1. *Let $G = H_{2m+1,2n+1}$ where $n-(m+1) = (2m+2)t+r$ and $0 \leq r \leq 2m+1$. If $r \in \{0, 1\}$, then $H_{2m+1,2n+1}$ is not dot-critical.*

Proof. Suppose that there exists an MDS S in $H_{2m+1,2n+1}$ such that $0, 2n \in S$. Since $\{0, 2n\}$ dominates $2m+3$ vertices, then $2n+1-(2m+3)$ vertices are not dominated by it.

Let $r = 0$. Since S is a dominating set in G and because $n \in S$ (by Lemma 4.1), it follows that $2t(2m+2) = 2n+1-(2m+3) \leq (2t-1)(2m+2)+1$, which is impossible.

Let $r = 1$. Because S is a dominating set in G and because $n \in S$, we deduce that
$2t(2m+2)+2 = 2n+1-(2m+3) \leq 2t(2m+2)+1$. This is a contradiction.

Finally, we show that both 0 and $2n$ are not critical. By the structure, it suffices to show that only one of them is not critical. Let S be an MDS of $H = G - 0$. Let $r = 0$. Since every vertex in H dominates at most $2m+2$ vertices and because $2n = (2t+1)(2m+2)$, we infer that $|S| \geq 2t+1 = \gamma(G)$. So, 0 is not critical. Now let $r = 1$. As every vertex in H dominates at most $2m+2$ vertices and since $2n = (2t+1)(2m+2)$, we deduce that $|S| \geq 2t+2 = \gamma(G)$. Therefore, 0 is not critical. Now Lemma 1.3 shows that G is not dot-critical. □

Theorem 4.2. *Let $G = H_{2m+1,2n+1}$. If $2n+1 = (2m+2)(2t+1)+2r+1$ where $2 \leq r \leq m+1$ and $t+r = m$, then G is $($totally$)$ dot-critical.*

Proof. It is immediate from Lemmas 1.2 (iii) and 1.3. □

Theorem 4.3. *Let $G = H_{2m+1,2n+1}$ with $2n+1 = (2m+2)(2t+1)+2r+1$. If $r \in \{m+2, \ldots, 2m+1\}$ or $t+r \neq m$, then no MDS contains both 0 and $2n$.*

Proof. Note that $2t+2 \leq \gamma(G) \leq 2t+3$. Suppose to the contrary that $0, 2n \in S$ for some MDS S in G. Note that the vertices 0 and $2n$ dominate $2m+3$ vertices. On the other hand, the number of non-dominated vertices by $\{0, 2n\}$ equals $2n+1-(2m+3) = (2m+2)2t+2r$. If $\gamma(G) = 2t+2$, then the number of vertices dominated by $S \setminus \{0, 2n\}$ at most equals $2t(2m+2)+1 \geq 2t(2m+2)+2r$, a contradiction. Let $\gamma(G) = 2t+3$. Invoking Lemma 1.1 (iii), this happens if and only if $r \geq m+2$. Therefore, the number of vertices dominated by $S \setminus \{0, 2n\}$ at most equals $(2t+1)(2m+2)+1 \geq 2t(2m+2)+2r \geq (2t+1)(2m+2)+2$, This presents a contradiction, thus concluding the proof. □

Combining Lemma 1.4 and Theorem 4.3 resulting the following theorem.

Theorem 4.4. *Let $G = H_{2m+1,2n+1}$ with $2n+1 = (2m+2)(2t+1)+2r+1$. If $r \in \{m+2, \ldots, 2m+1\}$ or $t+r \neq m$, then G is not $($totally$)$ dot-critical.*

5 Conclusions

Critical vertices play a crucial role in domination theory and have garnered significant attention in various associated research papers. Dot-critical graphs, a type of graph where contracting any edge leads to a decrease in the domination number, exhibit intriguing attributes linked to their critical vertices. This study delves into an exploration of the unique traits associated with dot-critical and totally dot-critical properties within the specific framework of Harary graphs. As this paper reaches its culmination, it is worth highlighting the importance of revisiting the inquiries previously posed in the work authored by Burton et al. This reflective approach contributes to a deeper understanding of the subject matter at hand [3].

1. What are the optimal diameter constraints for a k-dot-critical graph and a totally k-dot-critical graph G in the case where G is empty, and k is equal to or greater than 4?
2. Is the claim true that for every $k \geq 4$, there exists a k-totally dot-critical graph with no critical vertices?
3. Under what conditions does the critical domination of k-totally dot-critical graphs occur?

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